Labelled and Non-Wellfounded Proof Theory for Bimodal Provability Logic

Proof Society 2025 - Contributed Talk

Justus Becker

University of Birmingham

5 September 2025

Outline

1. Introduce basic modal provability logic (GL, axiomatisation, arithmetic/Kripke semantics)
2. Motivate non-wellfounded proofs for modal provability logic
3. Take a look into bimodal provability

Modal Provability Logic - The Basics

Provability as Modality

Let $\mathcal{L}^{\mathsf{EA}}$ be the first-order sentences of Elementary Arithmetic (EA) and $\mathsf{Prov}_{\mathsf{EA}}(\ulcorner \cdot \urcorner)$ some standard provability predicate of EA.

Provability as Modality

Let $\mathcal{L}^{\mathsf{EA}}$ be the first-order sentences of Elementary Arithmetic (EA) and $\mathsf{Prov}_{\mathsf{EA}}(\ulcorner\cdot\urcorner)$ some standard provability predicate of EA.

Let $\mathtt{Prop} = \{p, q, r, \ldots\}$ be a countable set of propositional atoms.

$$\mathcal{L}^{\Box} \ni A ::= \bot \mid p \in \mathtt{Prop} \mid A \to A \mid \Box A$$

Provability as Modality

Let $\mathcal{L}^{\mathsf{EA}}$ be the first-order sentences of Elementary Arithmetic (EA) and $\mathsf{Prov}_{\mathsf{EA}}(\ulcorner \cdot \urcorner)$ some standard provability predicate of EA.

Let $\mathtt{Prop} = \{p, q, r, ...\}$ be a countable set of propositional atoms.

$$\mathcal{L}^{\Box} \ni A ::= \bot \mid p \in \mathtt{Prop} \mid A \to A \mid \Box A$$

Provability interpretation $* : \mathcal{L}^{\Box} \to \mathcal{L}^{\mathsf{EA}}$ s.t.

1. $\bot^* = (0 = 1)$
2. $p^* \in \mathcal{L}^{\mathsf{EA}}$
3. $(A \to B)^* = A^* \to B^*$
4. $(\Box A)^* = \mathsf{Prov}_{\mathsf{EA}}(\ulcorner A^* \urcorner)$

Arithmetic Completeness

Gödel-Löb provability logic (GL):

CPL Classical Propositional Logic

K $\Box(A \to B) \to (\Box A \to \Box B)$

L $\Box(\Box A \to A) \to \Box A$

mp $\dfrac{A \quad A \to B}{B}$

nec $\dfrac{A}{\Box A}$

Arithmetic Completeness

Gödel-Löb provability logic (GL):

CPL Classical Propositional Logic

K $\Box(A \to B) \to (\Box A \to \Box B)$

L $\Box(\Box A \to A) \to \Box A$

mp $\dfrac{A \quad A \to B}{B}$

nec $\dfrac{A}{\Box A}$

Theorem (Löb 1955, Solovay 1976)

For all modal formulas A, $\mathsf{GL} \vdash A$ iff for all interpretations $*$: $\mathsf{EA} \vdash A^*$.

Kripke Semantics

A model for GL is a structure $\mathcal{M} = \langle W, \prec, V \rangle$ with
- a non-empty set of points $W \neq \emptyset$,
- a transitive, irreflexive and converse-wellfounded relation $\prec \subseteq W \times W$,
- and a valuation function $V : \texttt{Prop} \to \mathcal{P}(W)$.

$\mathcal{M}, x \not\vDash \bot$
$\mathcal{M}, x \vDash p$ iff $x \in V(p)$
$\mathcal{M}, x \vDash A \to B$ iff $\mathcal{M}, x \not\vDash A$ or $\mathcal{M}, x \vDash B$
$\mathcal{M}, x \vDash \Box A$ iff $\mathcal{M}, y \vDash A$ for all $y \in W$ with $x \prec y$

A formula is true on a model ($\mathcal{M} \vDash A$) if it is true on every point.

Theorem (Kripke Completeness, Segerberg 1971)

GL $\vdash A$ *iff for all* GL*-models* $\mathcal{M} \vDash A$.

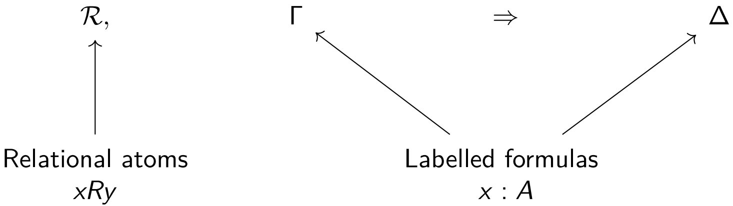
Proof Theory for Provability Logic

Labelled Sequents

Labelled proof systems can be thought of as formalising some parts of the semantics of a logic.
(They can also be seen as focussed systems for (fragments of) first-order logic.)

Labelled Sequents

Labelled proof systems can be thought of as formalising some parts of the semantics of a logic.
(They can also be seen as focussed systems for (fragments of) first-order logic.)



Labelled Sequents

Labelled proof systems can be thought of as formalising some parts of the semantics of a logic.
(They can also be seen as focussed systems for (fragments of) first-order logic.)

$$\mathcal{R}, \quad \Gamma \quad \Rightarrow \quad \Delta$$

Relational atoms
xRy

Labelled formulas
$x : A$

Proof search becomes searching for a countermodel, and thus failed proof search should always give us a countermodel.

Problem for GL:

Labelled Sequents

Labelled proof systems can be thought of as formalising some parts of the semantics of a logic.
(They can also be seen as focussed systems for (fragments of) first-order logic.)

$$\mathcal{R}, \quad \Gamma \quad \Rightarrow \quad \Delta$$

Relational atoms
xRy

Labelled formulas
$x : A$

Proof search becomes searching for a countermodel, and thus failed proof search should always give us a countermodel.

Problem for GL: Converse-wellfoundedness is a second-order property.

Sequent Calculus 1

The labelled system G3K4 due to Negri, 2005:

$$\text{Id}\ \frac{}{\mathcal{R}, \Gamma, x : p \Rightarrow x : p, \Delta} \qquad \bot\ \frac{}{\mathcal{R}, \Gamma, x : \bot \Rightarrow \Delta}$$

$$\to L\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \quad \mathcal{R}, \Gamma, x : B \Rightarrow \Delta}{\mathcal{R}, \Gamma, x : A \to B \Rightarrow \Delta} \qquad \to R\ \frac{\mathcal{R}, \Gamma, x : A \Rightarrow \Delta, x : B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \to B}$$

$$\Box L\ \frac{\mathcal{R}, xRy, x : \Box A, y : A, \Gamma \Rightarrow \Delta}{\mathcal{R}, xRy, x : \Box A, \Gamma \Rightarrow \Delta} \qquad \Box R\ \frac{\mathcal{R}, xRy, \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A}\ y!$$

$$trans\ \frac{\mathcal{R}, xRy, yRz, xRz, \Gamma \Rightarrow \Delta}{\mathcal{R}, xRy, yRz, \Gamma \Rightarrow \Delta}$$

A failed proof of Löb's axiom in G3K4

Non-Wellfounded Proofs

A failed proof of Löb's axiom in G3K4

$$\dfrac{}{\Rightarrow x : \Box(\Box p \to p) \to \Box p}$$

Non-Wellfounded Proofs

A failed proof of Löb's axiom in G3K4

$$\rightarrow R\ \dfrac{\overline{x : \Box(\Box p \rightarrow p) \Rightarrow x : \Box p}}{\Rightarrow x : \Box(\Box p \rightarrow p) \rightarrow \Box p}$$

Non-Wellfounded Proofs

A failed proof of Löb's axiom in G3K4

$$
\dfrac{\dfrac{\overline{xRy, x : \Box(\Box p \to p) \Rightarrow y : p}}{x : \Box(\Box p \to p) \Rightarrow x : \Box p}\,\Box R}{\Rightarrow x : \Box(\Box p \to p) \to \Box p}\,{\to}R
$$

Non-Wellfounded Proofs

A failed proof of Löb's axiom in G3K4

$$
\dfrac{\dfrac{\dfrac{\overline{xRy, x : \Box(\Box p \to p), y : \Box p \to p \Rightarrow y : p}}{xRy, x : \Box(\Box p \to p) \Rightarrow y : p}\,\Box L}{x : \Box(\Box p \to p) \Rightarrow x : \Box p}\,\Box R}{\Rightarrow x : \Box(\Box p \to p) \to \Box p}\,{\to}R
$$

Non-Wellfounded Proofs

A failed proof of Löb's axiom in G3K4

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overline{xRy, x : \Box(\Box p \to p) \Rightarrow y : p, y : \Box p} \quad \overline{xRy, x : \Box(\Box p \to p), y : p \Rightarrow y : p}}{xRy, x : \Box(\Box p \to p), y : \Box p \to p \Rightarrow y : p}\,{\to}L}{xRy, x : \Box(\Box p \to p) \Rightarrow y : p}\,\Box L}{x : \Box(\Box p \to p) \Rightarrow x : \Box p}\,\Box R}{\Rightarrow x : \Box(\Box p \to p) \to \Box p}\,{\to}R
$$

Non-Wellfounded Proofs

A failed proof of Löb's axiom in G3K4

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{xRy, x : \Box(\Box p \to p) \Rightarrow y : p, y : \Box p} \quad \overline{xRy, x : \Box(\Box p \to p), y : p \Rightarrow y : p}\,\text{Id}}{xRy, x : \Box(\Box p \to p), y : \Box p \to p \Rightarrow y : p}\,{\to}L}{xRy, x : \Box(\Box p \to p) \Rightarrow y : p}\,\Box L}{x : \Box(\Box p \to p) \Rightarrow x : \Box p}\,\Box R}{\Rightarrow x : \Box(\Box p \to p) \to \Box p}\,{\to}R$$

Non-Wellfounded Proofs

A failed proof of Löb's axiom in G3K4

$$
\rightarrow R\ \dfrac{\square R\ \dfrac{\square L\ \dfrac{\rightarrow L\ \dfrac{\square R\ \dfrac{\overline{xRy, yRz, x : \square(\square p \rightarrow p) \Rightarrow y : p, z : p}}{xRy, x : \square(\square p \rightarrow p) \Rightarrow y : p, y : \square p} \qquad \text{Id}\ \dfrac{}{xRy, x : \square(\square p \rightarrow p), y : p \Rightarrow y : p}}{xRy, x : \square(\square p \rightarrow p), y : \square p \rightarrow p \Rightarrow y : p}}{xRy, x : \square(\square p \rightarrow p) \Rightarrow y : p}}{x : \square(\square p \rightarrow p) \Rightarrow x : \square p}}{\Rightarrow x : \square(\square p \rightarrow p) \rightarrow \square p}
$$

A failed proof of Löb's axiom in G3K4

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\overline{xRy, yRz, xRz, x : \Box(\Box p \to p) \Rightarrow y : p, z : p}}{xRy, yRz, x : \Box(\Box p \to p) \Rightarrow y : p, z : p}\,\text{trans}
}{xRy, x : \Box(\Box p \to p) \Rightarrow y : p, y : \Box p}\,\Box R
\quad
\dfrac{}{xRy, x : \Box(\Box p \to p), y : p \Rightarrow y : p}\,\text{Id}
}{xRy, x : \Box(\Box p \to p), y : \Box p \to p \Rightarrow y : p}\,{\to}L
}{
\dfrac{xRy, x : \Box(\Box p \to p) \Rightarrow y : p}{
\dfrac{x : \Box(\Box p \to p) \Rightarrow x : \Box p}{\Rightarrow x : \Box(\Box p \to p) \to \Box p}\,{\to}R
}\,\Box R
}\,\Box L
$$

A failed proof of Löb's axiom in G3K4

$$
{\to}R\;\dfrac{\Box R\;\dfrac{\Box L\;\dfrac{{\to}L\;\dfrac{\Box R\;\dfrac{\mathit{trans}\;\dfrac{\Box L\;\dfrac{\vdots}{xRy, yRz, xRz, x : \Box(\Box p \to p) \Rightarrow y : p, z : p}}{xRy, yRz, x : \Box(\Box p \to p) \Rightarrow y : p, z : p}}{xRy, x : \Box(\Box p \to p) \Rightarrow y : p, y : \Box p} \qquad \mathrm{Id}\;\dfrac{}{xRy, x : \Box(\Box p \to p), y : p \Rightarrow y : p}}{xRy, x : \Box(\Box p \to p), y : \Box p \to p \Rightarrow y : p}}{xRy, x : \Box(\Box p \to p) \Rightarrow y : p}}{x : \Box(\Box p \to p) \Rightarrow x : \Box p}}{\Rightarrow x : \Box(\Box p \to p) \to \Box p}
$$

A failed proof of Löb's axiom in G3K4

$$
{\scriptstyle \to R}\dfrac{{\scriptstyle \Box R}\dfrac{{\scriptstyle \Box L}\dfrac{{\scriptstyle \to L}\dfrac{{\scriptstyle \Box R}\dfrac{{\scriptstyle trans}\dfrac{{\scriptstyle \Box L}\dfrac{\vdots}{xRy, yRz, xRz, x : \Box(\Box p \to p) \Rightarrow y : p, z : p}}{xRy, yRz, x : \Box(\Box p \to p) \Rightarrow y : p, z : p}}{xRy, x : \Box(\Box p \to p) \Rightarrow y : p, y : \Box p} \qquad {\scriptstyle \mathrm{Id}}\dfrac{}{xRy, x : \Box(\Box p \to p), y : p \Rightarrow y : p}}{xRy, x : \Box(\Box p \to p), y : \Box p \to p \Rightarrow y : p}}{xRy, x : \Box(\Box p \to p) \Rightarrow y : p}}{x : \Box(\Box p \to p) \Rightarrow x : \Box p}}{\Rightarrow x : \Box(\Box p \to p) \to \Box p}
$$

We end up in a loop! The proof failed and gives us a countermodel consisting of an infinite $\prec$-chain. **However** it is not a model of GL!

Sequent Calculus 2

In non-wellfounded and cyclic proof theory, proofs are no longer finite trees but can be infinite or may contain cycles.

The non-wellfounded and labelled calculus $\ell\mathsf{GL}^\infty$ due to Das, van der Giessen and Marin, 2023:

Definition (informal)

A *proof* in $\ell\mathsf{GL}^\infty$ is a possibly infinite tree (of labelled sequents) using the rules of G3K4 where every branch is either finite or has an always increasing R-chain.

Sequent Calculus 2

In non-wellfounded and cyclic proof theory, proofs are no longer finite trees but can be infinite or may contain cycles.

The non-wellfounded and labelled calculus $\ell\mathsf{GL}^\infty$ due to Das, van der Giessen and Marin, 2023:

Definition (informal)

A *proof* in $\ell\mathsf{GL}^\infty$ is a possibly infinite tree (of labelled sequents) using the rules of G3K4 where every branch is either finite or has an always increasing R-chain.

Theorem

For any formula $A \in \mathcal{L}^\Box$: $\mathsf{GL} \vdash A$ iff there is a proof of $\Rightarrow x : A$ in $\ell\mathsf{GL}^\infty$.

Bimodal Provability Logic

The basic logic CS

$$\mathcal{L}^{\Box\triangle} \ni A ::= \bot \mid p \in \texttt{Prop} \mid A \to A \mid \Box A \mid \triangle A$$

Let T and U be r.e. theories which can interpret EA, and let $(\Box A)^* = \mathrm{Prov}_T(\ulcorner A^* \urcorner)$ and $(\triangle A)^* = \mathrm{Prov}_U(\ulcorner A^* \urcorner)$. Then, the following principles hold (provably in EA).

The basic logic CS

$$\mathcal{L}^{\Box\triangle} \ni A ::= \bot \mid p \in \texttt{Prop} \mid A \to A \mid \Box A \mid \triangle A$$

Let T and U be r.e. theories which can interpret EA, and let $(\Box A)^* = \mathrm{Prov}_T(\ulcorner A^* \urcorner)$ and $(\triangle A)^* = \mathrm{Prov}_U(\ulcorner A^* \urcorner)$. Then, the following principles hold (provably in EA).

CPL	Classical Propositional Logic
$\mathsf{K}_\Box$	$\Box(A \to B) \to (\Box A \to \Box B)$
$\mathsf{L}_\Box$	$\Box(\Box A \to A) \to \Box A$
$\mathsf{K}_\triangle$	$\triangle(A \to B) \to (\triangle A \to \triangle B)$
$\mathsf{L}_\triangle$	$\triangle(\triangle A \to A) \to \triangle A$
$4_{\Box\triangle}$	$\triangle A \to \Box\triangle A$
$4_{\triangle\Box}$	$\Box A \to \triangle\Box A$

$$\mathsf{mp}\ \frac{A \to B \quad A}{B}$$

$$\mathsf{nec}_\Box\ \frac{A}{\Box A}$$

$$\mathsf{nec}_\triangle\ \frac{A}{\triangle A}$$

Carlson Models

A model for CS is a structure $\mathcal{M} = \langle W, \prec, M_0, M_1, V \rangle$ s.t. $\langle W, \prec, V \rangle$ is a GL-model and $M_0, M_1 \subseteq W$.

$$\mathcal{M}, x \vDash \Box A \text{ iff } \mathcal{M}, y \vDash A \text{ for all } y \in M_0 \text{ with } x \prec y$$

$$\mathcal{M}, x \vDash \triangle A \text{ iff } \mathcal{M}, y \vDash A \text{ for all } y \in M_1 \text{ with } x \prec y$$

Equivalently, we can define a model for CS by having two relations $\prec_\Box, \prec_\triangle$ s.t. $\prec_\Box \circ \prec_\triangle \subseteq \prec_\triangle$ and $\prec_\triangle \circ \prec_\Box \subseteq \prec_\Box$.

Sequent Calculus 3 (ℓCS^∞ / labCS$^\infty$)

$$\text{Id}\ \frac{}{\mathcal{R}, \Gamma, x : p \Rightarrow x : p, \Delta} \qquad \bot\ \frac{}{\mathcal{R}, \Gamma, x : \bot \Rightarrow \Delta}$$

$$\to L\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \quad \mathcal{R}, \Gamma, x : B \Rightarrow \Delta}{\mathcal{R}, \Gamma, x : A \to B \Rightarrow \Delta} \qquad \to R\ \frac{\mathcal{R}, \Gamma, x : A \Rightarrow \Delta, x : B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \to B}$$

$$\Box L\ \frac{\mathcal{R}, xRy, x : \Box A, y : A, \Gamma \Rightarrow \Delta}{\mathcal{R}, xRy, x : \Box A, \Gamma \Rightarrow \Delta} \qquad \Box R\ \frac{\mathcal{R}, xRy, \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A}\ y!$$

$$\triangle L\ \frac{\mathcal{R}, xSy, x : \triangle A, y : A, \Gamma \Rightarrow \Delta}{\mathcal{R}, xSy, x : \triangle A, \Gamma \Rightarrow \Delta} \qquad \triangle R\ \frac{\mathcal{R}, xSy, \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \triangle A}\ y!$$

$$trans_{RR}\ \frac{\mathcal{R}, xRy, yRz, xRz, \Gamma \Rightarrow \Delta}{\mathcal{R}, xRy, yRz, \Gamma \Rightarrow \Delta} \qquad trans_{SS}\ \frac{\mathcal{R}, xSy, ySz, xSz, \Gamma \Rightarrow \Delta}{\mathcal{R}, xSy, ySz, \Gamma \Rightarrow \Delta}$$

$$trans_{SR}\ \frac{\mathcal{R}, xSy, yRz, xRz, \Gamma \Rightarrow \Delta}{\mathcal{R}, xSy, yRz, \Gamma \Rightarrow \Delta} \qquad trans_{RS}\ \frac{\mathcal{R}, xRy, ySz, xSz, \Gamma \Rightarrow \Delta}{\mathcal{R}, xRy, ySz, \Gamma \Rightarrow \Delta}$$

Progress Condition $\approx$ Every infinite branch has an always increasing $R \cup S$-chain.

A Natural Bimodal Provability Logic

Let $(\Box A)^* = \mathsf{Prov}_{\mathsf{PA}}(\ulcorner A^* \urcorner)$ and $(\triangle A)^* = \mathsf{Prov}_{\mathsf{ZF}}(\ulcorner A^* \urcorner)$. We gain a maximal bimodal provability logic

$$\mathsf{ER} = \mathsf{CSM}_1 = \mathsf{CS} + \triangle(\Box A \to A)$$

A Natural Bimodal Provability Logic

Let $(\Box A)^* = \mathsf{Prov}_{\mathsf{PA}}(\ulcorner A^* \urcorner)$ and $(\triangle A)^* = \mathsf{Prov}_{\mathsf{ZF}}(\ulcorner A^* \urcorner)$. We gain a maximal bimodal provability logic

$$\mathsf{ER} = \mathsf{CSM}_1 = \mathsf{CS} + \triangle(\Box A \to A)$$

In a model of this logic $\langle W, R, M_0, M_1, V \rangle$ a point $x \in M_1$ looks something like this

$$y_1 \longleftarrow y_2 \longleftarrow y_3 \longleftarrow \cdots \cdots \quad x$$

such that there is an $n \in \omega$ s.t. for all $m \geq n$: $y_m \in V(p) \leftrightarrow x \in V(p)$.

A Natural Bimodal Provability Logic

Let $(\Box A)^* = \mathrm{Prov}_{\mathrm{PA}}(\ulcorner A^* \urcorner)$ and $(\triangle A)^* = \mathrm{Prov}_{\mathrm{ZF}}(\ulcorner A^* \urcorner)$. We gain a maximal bimodal provability logic

$$\mathrm{ER} = \mathrm{CSM}_1 = \mathrm{CS} + \triangle(\Box A \to A)$$

In a model of this logic $\langle W, R, M_0, M_1, V \rangle$ a point $x \in M_1$ looks something like this

$$y_1 \longleftarrow y_2 \longleftarrow y_3 \longleftarrow \cdots \cdots\cdots\cdots\cdots \; x$$

such that there is an $n \in \omega$ s.t. for all $m \geq n$: $y_m \in V(p) \leftrightarrow x \in V(p)$.

This calls for a different/new progress condition for a system for ER!

Takeaway

Labelled and non-wellfounded proof theory can give us a simple proof theoretic framework to build upon.
(Formalised first-order and (restricted) second-order reasoning.)

Takeaway

Labelled and non-wellfounded proof theory can give us a simple proof theoretic framework to build upon.
(Formalised first-order and (restricted) second-order reasoning.)

Future works:

- More extensions
- More modalities (e.g. GLP)
- Proof translations, e.g. sequentialise into unlabelled sequent systems

Q & A

Thank you for listening!