

# The Proof Theory and Semantics for Second-order (Intuitionistic) Tense Logic

British Colloquium for Theoretical Computer Science - Contributed Talk

Justus Becker, Anupam Das, Sonia Marin, Paaras Padhiar

University of Birmingham

1 April 2026



# Outline

- 1 Intuitionistic Modal Logic(s)
- 2 Second-Order Tense Logic
- 3 Semantics
- 4 Proof Theory
- 5 Soundness and Completeness
- 6 Conclusion

# Intuitionistic Modal Logic(s)

# Classical Modal Logic

Modalities:  $\Box \approx$  *necessary* and  $\Diamond \approx$  *possible*.

Classical basic modal logic K:

CPC Classical propositional logic

$$D_{\Box} \quad \Box(A \rightarrow B) \rightarrow \Box A \rightarrow \Box B$$

$$nec_{\Box} \quad \frac{\vdash A}{\Gamma \vdash \Box A}$$

$$\Box A \leftrightarrow \neg \Diamond \neg A$$

# Classical Modal Logic

Modalities:  $\Box \approx$  *necessary* and  $\Diamond \approx$  *possible*.

Classical basic modal logic K:

CPC Classical propositional logic

$$D_{\Box} \quad \Box(A \rightarrow B) \rightarrow \Box A \rightarrow \Box B$$

$$nec_{\Box} \quad \frac{\vdash A}{\Gamma \vdash \Box A}$$

$$\Box A \leftrightarrow \neg \Diamond \neg A$$

In intuitionistic logic, we do not want  $\Box A \leftrightarrow \neg \Diamond \neg A$

So, how to define  $\Diamond$  in intuitionistic logic?

# Intuitionistic Modal Logic(s)

|                      |   |   |    |   |    |
|----------------------|---|---|----|---|----|
| IPC                  | Intuitionistic Propositional Logic                                    | } | CK | } | IK |
| $nec_{\Box}$         | $\frac{\vdash A}{\Gamma \vdash \Box A}$                               |   |    |   |    |
| $D_{\Box}$           | $\Box(A \rightarrow B) \rightarrow \Box A \rightarrow \Box B$         |   |    |   |    |
| $D_{\Diamond}$       | $\Box(A \rightarrow B) \rightarrow \Diamond A \rightarrow \Diamond B$ |   |    |   |    |
| $N_{\Diamond \perp}$ | $\Diamond \perp \rightarrow \perp$                                    |   |    |   |    |
| $N_{\Diamond \vee}$  | $\Diamond(A \vee B) \rightarrow \Diamond A \vee \Diamond B$           |   |    |   |    |
| $I_{\Diamond \Box}$  | $(\Diamond A \rightarrow \Box B) \rightarrow \Box(A \rightarrow B)$   |   |    |   |    |

# Intuitionistic Modal Logic(s)

|                     |   |   |    |   |    |
|---------------------|---|---|----|---|----|
| IPC                 | Intuitionistic Propositional Logic                                    | } | CK | } | IK |
| $nec_{\Box}$        | $\frac{\vdash A}{\Gamma \vdash \Box A}$                               |   |    |   |    |
| $D_{\Box}$          | $\Box(A \rightarrow B) \rightarrow \Box A \rightarrow \Box B$         |   |    |   |    |
| $D_{\Diamond}$      | $\Box(A \rightarrow B) \rightarrow \Diamond A \rightarrow \Diamond B$ |   |    |   |    |
| $N_{\Diamond\perp}$ | $\Diamond\perp \rightarrow \perp$                                     |   |    |   |    |
| $N_{\Diamond\vee}$  | $\Diamond(A \vee B) \rightarrow \Diamond A \vee \Diamond B$           |   |    |   |    |
| $I_{\Diamond\Box}$  | $(\Diamond A \rightarrow \Box B) \rightarrow \Box(A \rightarrow B)$   |   |    |   |    |

These logics are well-studied and established:

Constructive K (CK) in Bellin et al. 2001

Intuitionistic K (IK) in Fischer Servi 1984 and Plotkin and Stirling 1986

# Intuitionistic Modal Logic(s)

|                      |   |   |    |   |    |
|----------------------|---|---|----|---|----|
| IPC                  | Intuitionistic Propositional Logic                                    | } | CK | } | IK |
| $nec_{\Box}$         | $\frac{\vdash A}{\Gamma \vdash \Box A}$                               |   |    |   |    |
| $D_{\Box}$           | $\Box(A \rightarrow B) \rightarrow \Box A \rightarrow \Box B$         |   |    |   |    |
| $D_{\Diamond}$       | $\Box(A \rightarrow B) \rightarrow \Diamond A \rightarrow \Diamond B$ |   |    |   |    |
| $N_{\Diamond \perp}$ | $\Diamond \perp \rightarrow \perp$                                    |   |    |   |    |
| $N_{\Diamond \vee}$  | $\Diamond(A \vee B) \rightarrow \Diamond A \vee \Diamond B$           |   |    |   |    |
| $I_{\Diamond \Box}$  | $(\Diamond A \rightarrow \Box B) \rightarrow \Box(A \rightarrow B)$   |   |    |   |    |

These logics are well-studied and established:

Constructive K (CK) in Bellin et al. 2001

Intuitionistic K (IK) in Fischer Servi 1984 and Plotkin and Stirling 1986

Note: Diamonds affect the behaviour of boxes!

$$CK \not\vdash \neg\neg\Box A \rightarrow \Box\neg\neg A \vdash IK$$

# Second-Order Tense Logic

# Impredicative Definition

In second-order intuitionistic propositional logic we allow quantification over *propositions*.

# Impredicative Definition

In second-order intuitionistic propositional logic we allow quantification over *propositions*.

This allows for *impredicativity*. E.g., in

$$\forall X(X \rightarrow P \rightarrow X)$$

we can instantiate the variable  $X$  with a more complex formula

$$\forall Y(\forall X(X \rightarrow (Y \rightarrow Y) \rightarrow X))$$

Giving us the following formula:

$$\forall Y(\forall X(X \rightarrow (Y \rightarrow Y) \rightarrow X)) \rightarrow P \rightarrow \forall Y(\forall X(X \rightarrow (Y \rightarrow Y) \rightarrow X))$$

# Impredicative Definition

In second-order intuitionistic propositional logic we allow quantification over *propositions*.

This allows for *impredicativity*. E.g., in

$$\forall X(X \rightarrow P \rightarrow X)$$

we can instantiate the variable  $X$  with a more complex formula

$$\forall Y(\forall X(X \rightarrow (Y \rightarrow Y) \rightarrow X))$$

Giving us the following formula:

$$\forall Y(\forall X(X \rightarrow (Y \rightarrow Y) \rightarrow X)) \rightarrow P \rightarrow \forall Y(\forall X(X \rightarrow (Y \rightarrow Y) \rightarrow X))$$

This also allows us to define all other logical connectives (using only  $\rightarrow$  and  $\forall$ ):  
E.g.

$$\forall X X := \perp \quad A \vee B := \forall X((A \rightarrow X) \rightarrow (B \rightarrow X) \rightarrow X)$$

# Defining Diamonds via impredicativity

Existential quantification can be defined as follows in second-order logic

$$\exists Y A := \forall X (\forall Y (A \rightarrow X) \rightarrow X)$$

# Defining Diamonds via impredicativity

Existential quantification can be defined as follows in second-order logic

$$\exists Y A := \forall X(\forall Y(A \rightarrow X) \rightarrow X)$$

Similarly we define the diamond modality by also admitting a backwards box  $\blacksquare$

$$\diamond A := \forall X(\Box(A \rightarrow \blacksquare X) \rightarrow X)$$

Symmetrically, we also define a backwards looking diamond

$$\blacklozenge A := \forall X(\blacksquare(A \rightarrow \Box X) \rightarrow X)$$

# Defining Diamonds via impredicativity

Existential quantification can be defined as follows in second-order logic

$$\exists Y A := \forall X(\forall Y(A \rightarrow X) \rightarrow X)$$

Similarly we define the diamond modality by also admitting a backwards box  $\blacksquare$

$$\diamond A := \forall X(\square(A \rightarrow \blacksquare X) \rightarrow X)$$

Symmetrically, we also define a backwards looking diamond

$$\blacklozenge A := \forall X(\blacksquare(A \rightarrow \square X) \rightarrow X)$$

This definition of dual modalities in second-order logic is one of the main novelties of this work.

From now on, we take diamonds to be defined as above.

# Second-Order Tense Logic

As we define  $\diamond$  and  $\blacklozenge$ , it is sufficient to consider the following language.

$$A ::= X \in Var \mid P \in Pr \mid A \rightarrow A \mid \forall X A \mid \square A \mid \blacksquare A$$

# Second-Order Tense Logic

As we define  $\diamond$  and  $\blacklozenge$ , it is sufficient to consider the following language.

$$A ::= X \in Var \mid P \in Pr \mid A \rightarrow A \mid \forall X A \mid \Box A \mid \blacksquare A$$

The logic IKt2 (intuitionistic tense second-order K) is then given by adding the following to second-order propositional logic:

$$\text{nec}_{\Box} \frac{\vdash A}{\Gamma \vdash \Box A}$$

$$\text{nec}_{\blacksquare} \frac{\vdash A}{\Gamma \vdash \blacksquare A}$$

$$D_{\Box} \Box(A \rightarrow B) \rightarrow \Box A \rightarrow \Box B$$

$$D_{\blacksquare} \blacksquare(A \rightarrow B) \rightarrow \blacksquare A \rightarrow \blacksquare B$$

$$A_{\Box, \diamond} A \rightarrow \Box \blacklozenge A$$

$$A_{\blacksquare, \blacklozenge} A \rightarrow \blacksquare \blacklozenge A$$

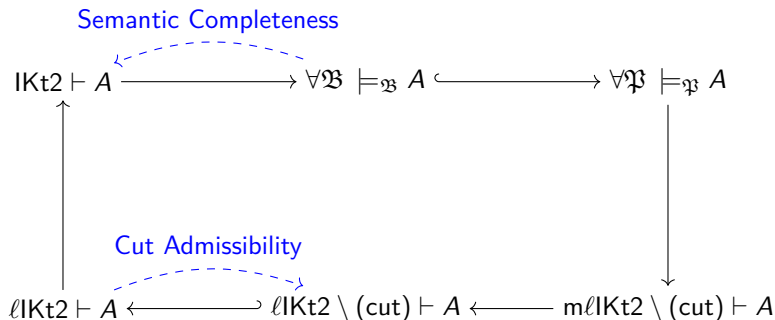
Importantly, IKt2 also contains the following comprehension axiom, which allows for impredicative reasoning:

$$\forall X A \rightarrow A[C/X]$$

# The Main Results

We show soundness and completeness of this logic with respect to

- $\mathfrak{B}$ irelational and  $\mathfrak{P}$ redicate semantics
- a cut-free proof system  $\ell\text{IKt2} \setminus (\text{cut})$  (thus also showing cut admissibility)



# A consequence of the main results

It follows from our result that IKt2 conservatively extends

- the intuitionistic logic IK (Fischer Servi 1984, Plotkin and Stirling 1986); and
- the tense intuitionistic logic IKt (Ewald 1986).

# Semantics

Second-order modal  $\mathfrak{P}$ predicate models are given (informally) by:

- a set of (intuitionistic) worlds  $\Omega$
  - a monotone pre-order  $\leq$  on  $\Omega$
- } intuitionistic Kripke model
- every world  $a \in \Omega$  yields a classical modal Kripke model consisting of
    - a (fixed) set of modal worlds  $W$
    - a modal relation  $R^a \subseteq W \times W$
    - a set of predicates  $\mathcal{W}^a \supseteq Pr$
    - an interpretation  $P^a \subseteq W$  for each  $P \in Pr$

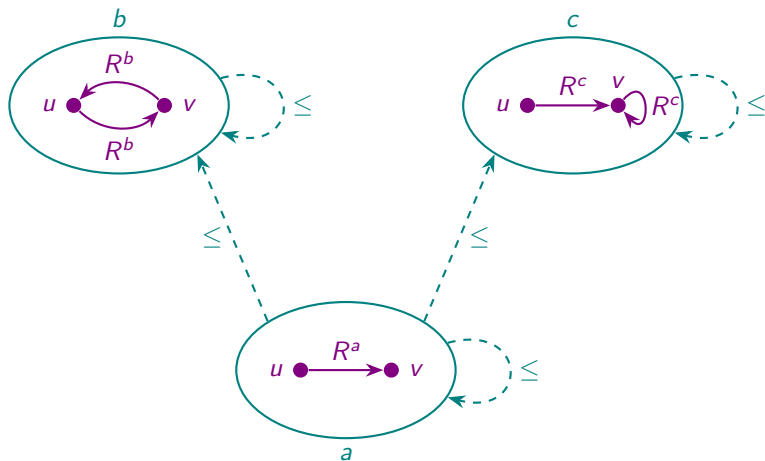
Second-order modal  $\mathfrak{P}$ predicate models are given (informally) by:

- a set of (intuitionistic) worlds  $\Omega$
  - a monotone pre-order  $\leq$  on  $\Omega$
  - every world  $a \in \Omega$  yields a **classical modal Kripke model** consisting of
    - a (fixed) set of modal worlds  $W$
    - a modal relation  $R^a \subseteq W \times W$
    - a set of predicates  $\mathcal{W}^a \supseteq Pr$
    - an interpretation  $P^a \subseteq W$  for each  $P \in Pr$
- } intuitionistic Kripke model

Two things to note about the set of predicates  $\mathcal{W}^a$ :

- 1  $\mathcal{W}^a$  restricts the second-order quantification, yielding a Henkin semantics
- 2  $\mathcal{W}^a$  should be *comprehensive*: i.e. for every formula  $C$ , we want a corresponding predicate  $[C] \in \mathcal{W}^a$

# Example Model



We evaluate (closed) formulas locally:

- $\mathfrak{F}, u, a \models P$  iff  $u \in P^a$
- $\mathfrak{F}, u, a \models A \rightarrow B$  iff  $\forall b \geq a$  if  $\mathfrak{F}, u, b \models A$  then  $\mathfrak{F}, u, b \models B$
- $\mathfrak{F}, u, a \models \forall X A$  iff  $\forall b \geq a \forall V \in \mathcal{W}$  we have  $\mathfrak{F}, u, b \models A[V/X]$
- $\mathfrak{F}, u, a \models \Box A$  iff  $\forall b \geq a \forall v \in W$  if  $uR^b v$  then  $\mathfrak{F}, v, b \models A$
- $\mathfrak{F}, u, a \models \blacksquare A$  iff  $\forall b \geq a \forall v \in W$  if  $vR^b u$  then  $\mathfrak{F}, v, b \models A$

We say  $\mathfrak{F} \models A$  if  $\mathfrak{F}, u, a \models A$  for all  $a \in \Omega$  and  $u \in W$ .

# Semantic Soundness

We evaluate (closed) formulas locally:

- $\mathfrak{F}, u, a \models P$  iff  $u \in P^a$
- $\mathfrak{F}, u, a \models A \rightarrow B$  iff  $\forall b \geq a$  if  $\mathfrak{F}, u, b \models A$  then  $\mathfrak{F}, u, b \models B$
- $\mathfrak{F}, u, a \models \forall X A$  iff  $\forall b \geq a \forall V \in \mathcal{W}$  we have  $\mathfrak{F}, u, b \models A[V/X]$
- $\mathfrak{F}, u, a \models \Box A$  iff  $\forall b \geq a \forall v \in W$  if  $uR^b v$  then  $\mathfrak{F}, v, b \models A$
- $\mathfrak{F}, u, a \models \blacksquare A$  iff  $\forall b \geq a \forall v \in W$  if  $vR^b u$  then  $\mathfrak{F}, v, b \models A$

We say  $\mathfrak{F} \models A$  if  $\mathfrak{F}, u, a \models A$  for all  $a \in \Omega$  and  $u \in W$ .

We can view  $\mathfrak{F}$  predicate models as more general  $\mathfrak{B}$ irelational models by essentially identifying pairs  $(u, a)$  as single worlds.

$$\forall \mathfrak{B} \models_{\mathfrak{B}} A \longleftarrow \longrightarrow \forall \mathfrak{F} \models_{\mathfrak{F}} A$$

# Semantic Soundness

We evaluate (closed) formulas locally:

- $\mathfrak{F}, u, a \models P$  iff  $u \in P^a$
- $\mathfrak{F}, u, a \models A \rightarrow B$  iff  $\forall b \geq a$  if  $\mathfrak{F}, u, b \models A$  then  $\mathfrak{F}, u, b \models B$
- $\mathfrak{F}, u, a \models \forall X A$  iff  $\forall b \geq a \forall V \in \mathcal{W}$  we have  $\mathfrak{F}, u, b \models A[V/X]$
- $\mathfrak{F}, u, a \models \Box A$  iff  $\forall b \geq a \forall v \in W$  if  $uR^b v$  then  $\mathfrak{F}, v, b \models A$
- $\mathfrak{F}, u, a \models \blacksquare A$  iff  $\forall b \geq a \forall v \in W$  if  $vR^b u$  then  $\mathfrak{F}, v, b \models A$

We say  $\mathfrak{F} \models A$  if  $\mathfrak{F}, u, a \models A$  for all  $a \in \Omega$  and  $u \in W$ .

We can view  $\mathfrak{F}$  predicate models as more general  $\mathfrak{B}$ irelational models by essentially identifying pairs  $(u, a)$  as single worlds.

By a standard soundness argument we also have:

$$\text{IKt2} \vdash A \longrightarrow \forall \mathfrak{B} \models_{\mathfrak{B}} A \longleftarrow \forall \mathfrak{F} \models_{\mathfrak{F}} A$$

# Proof Theory

# Labelled Proof Theory

By the standard translation, we can read formulas of IKt2 as formulas of intuitionistic second-order *predicate* logic.

# Labelled Proof Theory

By the standard translation, we can read formulas of IKt2 as formulas of intuitionistic second-order *predicate* logic.

This leads to a natural sequent calculus for IKt2, using labels  $u, v, w, \dots$  and labelled sequents

$$\mathcal{R} \mid \Gamma \Rightarrow w : A$$

with  $\mathcal{R}$  containing relational atoms  $uRv$  and  $\Gamma$  containing labelled formulas  $w : A$ .

Example rules:

$$\Box_r \frac{\mathcal{R}, vRu \mid \Gamma \Rightarrow u : A}{\mathcal{R} \mid \Gamma \Rightarrow v : \Box A} \quad u \text{ fresh}$$

$$\Box_l \frac{\mathcal{R}, uRv \mid \Gamma, u : A \Rightarrow w : B}{\mathcal{R}, uRv \mid \Gamma, v : \blacksquare A \Rightarrow w : B}$$

$$\forall_l \frac{\mathcal{R} \mid \Gamma, v : A[C/X] \Rightarrow w : B}{\mathcal{R} \mid \Gamma, v : \forall X A \Rightarrow w : B} \quad \rightsquigarrow \text{ note that } C \text{ might be a larger formula}$$

We call this calculus (together with the cut rule)  $\ell$ IKt2.

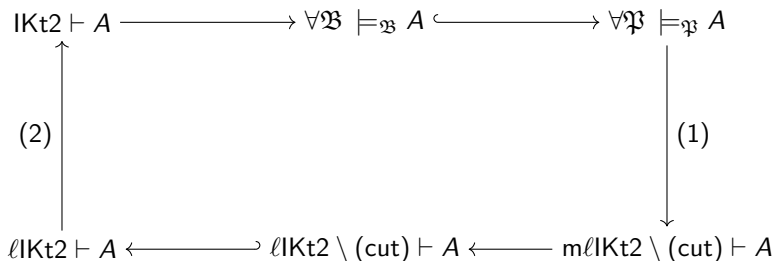


# Soundness and Completeness

# Going via the Grand Tour

We show

- (1) completeness of  $\ell IKt2$  with respect to the semantics.
- (2) soundness with respect to the Hilbert axiomatisation and



# The issue of standard Cut-Elimination

$$\frac{\frac{\vdots}{\mathcal{R} \mid \Gamma \Rightarrow u : P \rightarrow Q} \quad \frac{\vdots}{\mathcal{R}' \mid \Gamma', u : C \rightarrow Q \Rightarrow w : B}}{\frac{\forall_r \mathcal{R} \mid \Gamma \Rightarrow u : \forall X(X \rightarrow Q) \quad \forall_l \mathcal{R}' \mid \Gamma', u : \forall X(X \rightarrow Q) \Rightarrow w : B}{\mathcal{R}, \mathcal{R}' \mid \Gamma, \Gamma' \Rightarrow w : B}} \text{(cut)}$$

Now, we would like to cut  $u : P \rightarrow Q$  with  $u : C \rightarrow Q$ . However,  $C$  might be a complex formula. Thus, this step might not reduce the rank of the cut.

# Proof Search

We work in a slightly adapted but equivalent system,  $m\ell IKt2 \setminus (\text{cut})$ , which makes the counter-model construction easier.

# Proof Search

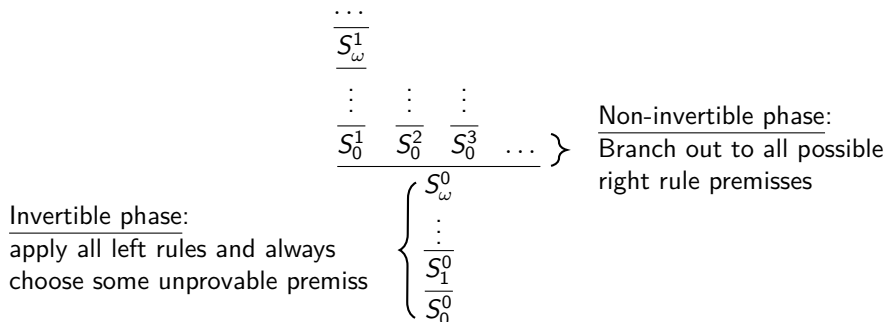
We work in a slightly adapted but equivalent system,  $m\ell IKt2 \setminus (\text{cut})$ , which makes the counter-model construction easier.

Starting with an unprovable formula  $m\ell IKt2 \setminus (\text{cut}) \not\vdash A$ , we construct a proof-search tree.

# Proof Search

We work in a slightly adapted but equivalent system,  $m\ell IKt2 \setminus (\text{cut})$ , which makes the counter-model construction easier.

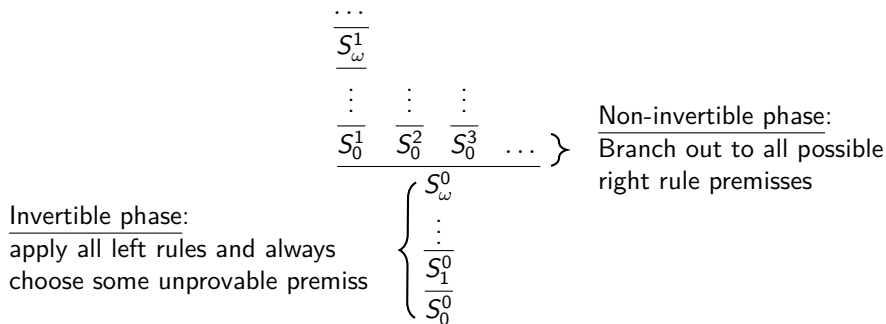
Starting with an unprovable formula  $m\ell IKt2 \setminus (\text{cut}) \not\vdash A$ , we construct a proof-search tree.



# Proof Search

We work in a slightly adapted but equivalent system,  $m\ell IKt2 \setminus (\text{cut})$ , which makes the counter-model construction easier.

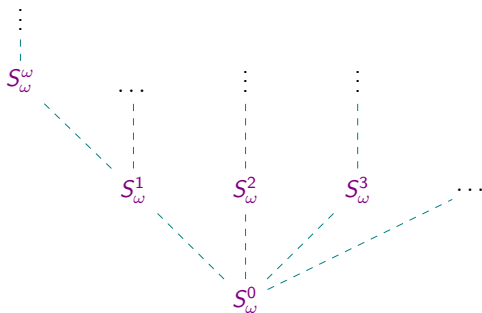
Starting with an unprovable formula  $m\ell IKt2 \setminus (\text{cut}) \not\vdash A$ , we construct a proof-search tree.



Note that the limit sequents  $S_{\omega}^0, S_{\omega}^1, \dots$  might contain infinitely many formulas. We therefore work with sequents of possibly infinite size.

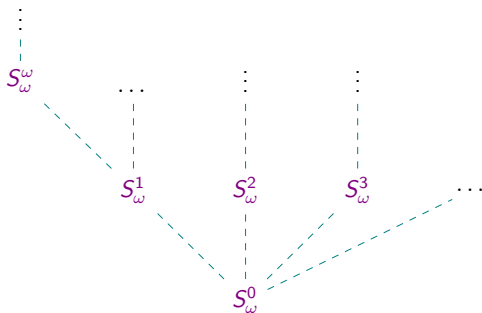
# From Pre-Model to full Countermodel I

Consider the tranfinite tree of all limit sequents of (i.e. the last sequents of the invertible phases) all of which are unprovable:



# From Pre-Model to full Countermodel I

Consider the tranfinite tree of all limit sequents of (i.e. the last sequents of the invertible phases) all of which are unprovable:



This yields a pre-model

- the limit sequents  $S_\omega^\alpha$  constitute (intuitionistic) worlds, connected by the branching of the non-invertible phases  $(\Omega, \leq)$
- the *labels and relational atoms* of the sequent  $S_\omega^\alpha$  constitute modal worlds and relations  $(W, R^a)$

# From Pre-Model to full Countermodel II

To define the missing information for the model, we use *possible extensions*.  
Informally, a possible extension  $C$  for  $S_\omega^\alpha$  is a set of worlds which might satisfy  $C$  without validating  $S_\omega^\alpha$ .

# From Pre-Model to full Countermodel II

To define the missing information for the model, we use *possible extensions*. Informally, a possible extension  $C$  for  $S_\omega^\alpha$  is a set of worlds which might satisfy  $C$  without validating  $S_\omega^\alpha$ .

- We define  $\mathcal{W}^a$  to be the set of all *possible extensions*  $C$  for every formula  $C$  for  $a = S_\omega^\alpha$ .
- We define the interpretation  $P^a$  of  $P \in \text{Pr} \subseteq \mathcal{W}^a$  as its smallest possible extension.

# From Pre-Model to full Countermodel II

To define the missing information for the model, we use *possible extensions*. Informally, a possible extension  $C$  for  $S_\omega^\alpha$  is a set of worlds which might satisfy  $C$  without validating  $S_\omega^\alpha$ .

- We define  $\mathcal{W}^a$  to be the set of all *possible extensions*  $C$  for every formula  $C$  for  $a = S_\omega^\alpha$ .
- We define the interpretation  $P^a$  of  $P \in \text{Pr} \subseteq \mathcal{W}^a$  as its smallest possible extension.

We can now show via impredicativity that: **possible extensions are compatible with their respective formulas.**

Crucially, we want that if  $\forall X A(X)$  should be true in  $S_\omega^\alpha$  we also have that  $A(C)$  is true at  $S_\omega^\alpha$  in our model (for every formula  $C$  and any of its extensions  $C$ ).

# From Pre-Model to full Countermodel II

To define the missing information for the model, we use *possible extensions*. Informally, a possible extension  $C$  for  $S_\omega^\alpha$  is a set of worlds which might satisfy  $C$  without validating  $S_\omega^\alpha$ .

- We define  $\mathcal{W}^a$  to be the set of all *possible extensions*  $C$  for every formula  $C$  for  $a = S_\omega^\alpha$ .
- We define the interpretation  $P^a$  of  $P \in \text{Pr} \subseteq \mathcal{W}^a$  as its smallest possible extension.

We can now show via impredicativity that: **possible extensions are compatible with their respective formulas.**

Crucially, we want that if  $\forall X A(X)$  should be true in  $S_\omega^\alpha$  we also have that  $A(C)$  is true at  $S_\omega^\alpha$  in our model (for every formula  $C$  and any of its extensions  $C$ ).

As it turns out, the interpretation of a formula  $[C]$  is always a possible extension. Thus, we also have comprehensiveness of the model.

# From Pre-Model to full Countermodel II

To define the missing information for the model, we use *possible extensions*. Informally, a possible extension  $C$  for  $S_\omega^\alpha$  is a set of worlds which might satisfy  $C$  without validating  $S_\omega^\alpha$ .

- We define  $\mathcal{W}^a$  to be the set of all *possible extensions*  $C$  for every formula  $C$  for  $a = S_\omega^\alpha$ .
- We define the interpretation  $P^a$  of  $P \in \text{Pr} \subseteq \mathcal{W}^a$  as its smallest possible extension.

We can now show via impredicativity that: **possible extensions are compatible with their respective formulas.**

Crucially, we want that if  $\forall X A(X)$  should be true in  $S_\omega^\alpha$  we also have that  $A(C)$  is true at  $S_\omega^\alpha$  in our model (for every formula  $C$  and any of its extensions  $C$ ).

As it turns out, the interpretation of a formula  $[C]$  is always a possible extension. Thus, we also have comprehensiveness of the model.

Similar semantic completeness arguments for second-order logic can also be found in Prawitz 1968 and Tait 1966.

# Grand Tour Revisited

Until now, we have the following:

$$\begin{array}{ccccc} \text{IKt2} \vdash A & \longrightarrow & \forall \mathfrak{B} \models_{\mathfrak{B}} A & \longleftarrow & \forall \mathfrak{P} \models_{\mathfrak{P}} A \\ & & & & \downarrow \\ \ell\text{IKt2} \vdash A & \longleftarrow & \ell\text{IKt2} \setminus (\text{cut}) \vdash A & \longleftarrow & \text{m}\ell\text{IKt2} \setminus (\text{cut}) \vdash A \end{array}$$

# Grand Tour Revisited

Until now, we have the following:

$$\begin{array}{ccccc} \text{IKt2} \vdash A & \longrightarrow & \forall \mathfrak{B} \models_{\mathfrak{B}} A & \longleftarrow & \forall \mathfrak{B} \models_{\mathfrak{B}} A \\ & & & & \downarrow \\ \ell\text{IKt2} \vdash A & \longleftarrow & \ell\text{IKt2} \setminus (\text{cut}) \vdash A & \longleftarrow & \text{m}\ell\text{IKt2} \setminus (\text{cut}) \vdash A \end{array}$$

To show the final missing arrow, we emulate rules of  $\ell\text{IKt2}$  via derivations in the Hilbert calculus  $\text{IKt2}$ .

We interpret labelled sequents  $\mathcal{R} \mid \Gamma \Rightarrow u : C$  as formulas  $\text{fm}(\mathcal{R} \mid \Gamma \Rightarrow u : C)$ .

We interpret labelled sequents  $\mathcal{R} \mid \Gamma \Rightarrow u : C$  as formulas  $\text{fm}(\mathcal{R} \mid \Gamma \Rightarrow u : C)$ .

By showing that for every rule in  $\ell\text{IKt2}$

$$\frac{S_1 \quad \dots \quad S_n}{S}$$

its translation

$$\frac{\text{fm}(S_1) \quad \dots \quad \text{fm}(S_n)}{\text{fm}(S)}$$

is derivable in the Hilbert calculus  $\text{IKt2}$ , we get:

If  $\ell\text{IKt2} \vdash S$  then  $\text{IKt2} \vdash \text{fm}(S)$ .

We interpret labelled sequents  $\mathcal{R} \mid \Gamma \Rightarrow u : C$  as formulas  $\text{fm}(\mathcal{R} \mid \Gamma \Rightarrow u : C)$ .

By showing that for every rule in  $\ell\text{IKt2}$

$$\frac{S_1 \quad \dots \quad S_n}{S}$$

its translation

$$\frac{\text{fm}(S_1) \quad \dots \quad \text{fm}(S_n)}{\text{fm}(S)}$$

is derivable in the Hilbert calculus  $\text{IKt2}$ , we get:

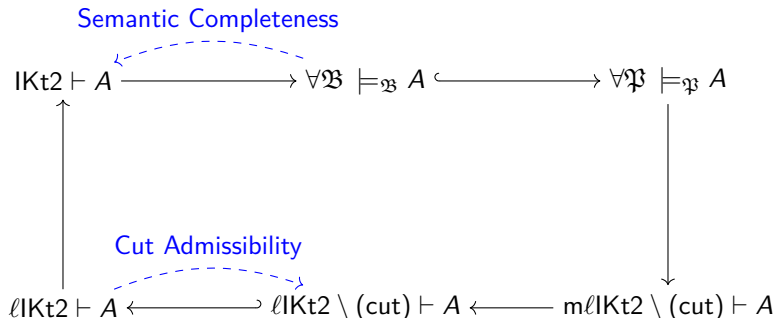
If  $\ell\text{IKt2} \vdash S$  then  $\text{IKt2} \vdash \text{fm}(S)$ .

By  $\text{fm}(\mid \Rightarrow u : A) = A$ , we obtain as a corollary:

If  $\ell\text{IKt2} \vdash A$  then  $\text{IKt2} \vdash A$ .

# The Grand Tour

We have therefore shown that all the following characterise the same logic.



# Conclusion

What we did in this work:

- provide axiomatisation, semantics and proof theory of a minimal second-order tense logic
- prove soundness and completeness (including cut admissibility)
- show that our logic  $\text{IKt2}$  conservatively extends the tense logic  $\text{IKt}$  and the modal logic  $\text{IK}$
- specialised the results to the classical case where we obtain a classical second-order tense logic  $\text{Kt2}$

There are two main research directions going further:

- define (intuitionistic) fixpoint operators directly by utilising second-order quantification together with a global modality ( $\mathbb{G}$  = “everywhere”)

$$\mu X A(X) := \forall X (\mathbb{G}(A(X) \rightarrow X) \rightarrow X)$$

- apply the Curry-Howard perspective of proofs-as-programs to IKt2, including the development of a second-order modal type theory and a natural deduction calculus

Thank you!  
Questions?