▸ JUSTUS BECKER, REVANTHA RAMANAYAKE, *Towards interpolation with non-analytic cuts using strong Lyndon interpolants.*
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A propositional logic has the *Craig interpolation property* [2] if for any provable implication $A \to B$, there is a Craig interpolant I which sits between A and B in the following sense:

1. I contains only propositional atoms that are common to A and B;
2. $A \to I$ is provable; and
3. $I \to B$ is provable.

These conditions are also referred to as the *variable condition* (1.) and the *provability conditions* (2.-3.).

A standard way to show interpolation for a logic, known as *Maehara's method* [6], is via a suitable cut-free sequent system. Beginning with a proof of an implication $A \to B$, we can determine a partition for every sequent $\Gamma; \Gamma' \Rightarrow C$, where one side of the partition corresponds to A while the other side corresponds to B. Then, one constructs the interpolant inductively going top-down through the proof depending on the rules and the partition.

Using *Mints' method* [8], this approach has been expanded further towards systems which do not have cut-elimination but still are analytic by virtue of analytic cuts [9, 11]. The local analyticity of cut is crucial there, as we can view the cut formula occurring in the premisses of a cut as belonging to that side of the partition of which it is a subformula. Thus, the approach of Mints seems inappropriate for non-analytic cuts. In this work in progress, we pursue the question on how we can push cuts beyond analyticity while still obtaining interpolation results.

Let us consider the intermediate logic $\mathsf{GD} = \mathsf{IPL} + (A \to B) \lor (B \to A)$ known as Gödel-Dummett logic or fuzzy logic as the next non-trivial case. It is well known that GD has the Craig interpolation property [7], and while there are analytic sequent systems for this logic [1, 3], they do not allow us to apply Maehara's method directly. Further, such an approach would not necessarily be generalisable to other logics such as three-valued Gödel logic G_3 or the intermediate logic of maximal bounded depth two BD_2 which are also known to have Craig interpolation [7].

Previously, the proof-theoretic way to deal with interpolation for such logics has been to extend the notion of sequents and interpolants towards higher structures such as hypersequents [4]. Here, we want to stay close to the original idea of Maehara's method to gain a simple sequent based proof that avoids semantic arguments, thus also developing general techniques for proof theory with restricted cuts. By [10], we know that the logic GD can be axiomatised over IPL with only atomic instances of $\neg p \lor \neg\neg p$ and $(p \to q) \lor ((p \to q) \to p)$. We can therefore obtain a sound and complete sequent system by adding the following cut-rules to the intuitionistic sequent system LJ:

$$\dfrac{\dfrac{}{\Rightarrow \neg p \lor \neg\neg p}\,wlem \qquad \neg p \lor \neg\neg p, \Gamma \Rightarrow A}{\Gamma \Rightarrow A}\,(\mathrm{cut})$$

$$\dfrac{\dfrac{}{\Rightarrow (p \to q) \lor ((p \to q) \to p)}\,lin \qquad (p \to q) \lor ((p \to q) \to p), \Gamma \Rightarrow A}{\Gamma \Rightarrow A}\,(\mathrm{cut})$$

For showing Craig interpolation with these cuts present, we need to choose a partition of the right upper sequent depending on the lower sequent, just like one does for the operational rules. Ideally, we want to choose the partition of the right upper sequents so that its interpolant also works as an interpolant for the lower sequent. Note that here, the only thing to consider is the variable condition as both *wlem* and *lin* are equivalent to $\top$ in GD and thus do not impact provability when occurring in the antecedent.

In the case of cutting with weak excluded middle (*wlem*), we are given a partition $\Gamma;\Gamma' \Rightarrow A$ of the lower sequent and want to find a "good" partition for $\neg p \vee \neg\neg p, \Gamma, \Gamma' \Rightarrow A$. This is unproblematic: if the atom p occurs on both sides of the partition (i.e. both Γ and $\Gamma' \Rightarrow A$) or it occurs in neither side, the additional p instances from the weak excluded middle do not impact the variable condition of the interpolant. If p occurs only in one side we can choose the cut formula to be in exactly that side of the partition. In this case, we obtain an interpolant I which does not contain the atom p, thus it is also an interpolant for the lower sequent as I is "blind" to any instances of p.

For instances of *lin*-cut, there is still an obstacle, as we might find p and q on different sides of the partition. For example, for a given partition $\Gamma;\Gamma' \Rightarrow A$ we might find that p occurs only in Γ while q occurs only in $\Gamma' \Rightarrow A$. Thus, an interpolant for $\Gamma;\Gamma' \Rightarrow A$ should not neither p nor q. However, for both the partitions $(p \to q) \vee ((p \to q) \to p), \Gamma; \Gamma' \Rightarrow A$ and $\Gamma; (p \to q) \vee ((p \to q) \to p), \Gamma' \Rightarrow A$ we might find an interpolant which contains either a p or a q. Let us simplify the situation in multiple steps.

One way to simplify the problem is by noticing that we can always invertibly apply $\vee_L$ above a cut instance, thus essentially allowing us to obtain the following gadget:

$$\frac{p \to q, \Gamma, \Gamma' \Rightarrow A \quad (p \to q) \to p, \Gamma, \Gamma' \Rightarrow A}{\Gamma; \Gamma' \Rightarrow A}\, {\scriptstyle lin(\mathrm{cut})\vee_L}$$

While initially this seems like we did not gain much, we now have separated atoms of different *polarity*. To see how this helps, consider as an example the case where Γ has a positive occurrence of p while $\Gamma' \Rightarrow A$ contains a negative p and a negative q. Notice that in the partition $\Gamma; (p \to q) \to p, \Gamma' \Rightarrow A$ the atom p occurs only positively on the left and only negatively on the right while q occurs only on the right. Thus, a *Lyndon* interpolant [5] for this partition contains neither p nor q, as the variable condition is restricted to atoms of the same polarity. A Lyndon interpolant for $\Gamma; (p \to q) \to p, \Gamma' \Rightarrow A$ will therefore also be an interpolant for $\Gamma; \Gamma' \Rightarrow A$. Notice that we gain quite some flexibility, as in some cases it is enough to find a partition for one of the premisses of the gadget.

But even with this gadget, there are cases where neither upper sequent can guarantee us such an interpolant. In these cases, however, we do not actually get a "bad" interpolant assuming a reasonable interpolation construction. Informally, this is because the interpolant would contain a propositional atom which is actually not needed for the provability condition. Consider for example, a formula $A = q \vee (p \to r)$ and a formula I containing a negative instance of p such that $A \to I$ is provable. Then, we can see by reasoning about proof search that I should also contain a positive occurrence of r.

To accommodate for the idea that interpolants should be "reasonable" in the way we just described, we propose a notion of interpolant, which we call *strong Lyndon interpolant*. The intuitive idea is that such an interpolant keeps track of how atoms are related via implications in the sequent. However, the precise definition of strong Lyndon interpolant is still part of our ongoing research. Assuming that we can transfer strong Lyndon interpolants down operational rules, we can also transfer them down applications of *lin*-cuts. This will allow us to gain a new proof of interpolation for GD as well hopefully lead to similar results for other intermediate logic such as G_3 and BD_2.

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